

Weak measurements are universal

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It is well known that any projective measurement can be decomposed into a sequence of *weak* measurements, which cause only small changes to the state. Similar constructions for generalized measurements, however, have relied on the use of an ancilla system. We show that any generalized measurement can be decomposed into a sequence of weak measurements without the use of an ancilla, and give an explicit construction for these weak measurements. The measurement procedure has the structure of a random walk along a curve in state space, with the measurement ending when one of the end points is reached. This shows that any measurement can be generated by weak measurements, and hence that weak measurements are universal. This may have important applications to the theory of entanglement.

In the original formulation of measurement in quantum mechanics, measurement outcomes are identified with a set of orthogonal projection operators, which can be thought of as corresponding to the eigenspaces of a Hermitian operator, or *observable* [1, 2]. After a measurement, the state is projected into one of the subspaces with a probability given by the square of the amplitude of the state component in that subspace.

In recent years a more general notion of measurement has become common: the *generalized* or *positive-operator valued* measurement (POVM) [3]. This formulation can include many phenomena not captured by projective measurements: detectors with non-unit efficiency, measurement outcomes that include additional randomness, measurements that give incomplete information, and many others. POVMs have found numerous applications, especially in the rapidly-growing field of quantum information processing [4].

Upon measurement, a system with density matrix ρ undergoes a random transformation

$$\rho \rightarrow \rho_j = \hat{M}_j \rho \hat{M}_j^\dagger / p_j, \quad \sum_j \hat{M}_j^\dagger \hat{M}_j = \hat{I}, \quad (1)$$

with probability $p_j = \text{Tr}(\hat{M}_j \rho \hat{M}_j^\dagger)$, where the index j labels the possible outcomes of the measurement. This transformation is commonly comprehended as a spontaneous *jump*, unlike unitary transformations, for example, which are thought of as resulting from *continuous* unitary evolutions. Any unitary transformation can be implemented as a sequence of *weak* (i.e., infinitesimal) unitary transformations. One may ask if a similar decomposition exists for generalized measurements. This would allow us to think of POVMs as resulting from continuous stochastic evolutions and possibly make use of the powerful tools of differential calculus in the study of the transformations

that a system undergoes upon measurement.

In this paper we show that any generalized measurement can be implemented as a sequence of *weak measurements* [5, 6]. We call a measurement *weak* if all outcomes result in very small changes to the state. (There are other definitions of weak measurements that include the possibility of large changes to the state with low probability; we will not be considering measurements of this type.) Therefore, a weak measurement is one whose operators can be written as

$$\hat{M}_j = q_j(\hat{I} + \hat{\varepsilon}_j), \quad (2)$$

where $0 \leq q_j \leq 1$ and $\hat{\varepsilon}$ is an operator with small norm $\|\hat{\varepsilon}\| \ll 1$. Weak measurements have been studied both in the abstract, and as a means of understanding systems with continuous monitoring. In the latter case, we can think of the evolution as the limit of a sequence of weak measurements, which gives rise to continuous stochastic evolutions called *quantum trajectories* [6, 7].

It has been shown that any projective measurement can be done as a sequence of weak measurements; and by using an additional *ancilla* system and a joint unitary transformation, it is possible to do any generalized measurement using weak measurements [8]. This procedure, however, does not decompose the operation on the original system into weak operations, since it uses operations acting on a larger Hilbert space—that of the system plus the ancilla. If we wish to study the behavior of a function—for instance, an entanglement monotone—defined on a space of a particular dimension, it complicates matters to add and remove ancillas. We will show that an ancilla is not needed, and give an explicit construction of the weak measurement operators for any generalized measurement that we wish to decompose.

It is easy to show that a measurement with any number

of outcomes can be performed as a sequence of measurements with two outcomes. Therefore, for simplicity, we will restrict our considerations to two-outcome measurements. To give the idea of the construction, we first show how every projective measurement can be implemented as a sequence of weak generalized measurements. In this case the measurement operators \hat{P}_1 and \hat{P}_2 are orthogonal projectors whose sum $\hat{P}_1 + \hat{P}_2 = \hat{I}$ is the identity. We introduce the operators

$$\hat{P}(x) = \sqrt{\frac{1 - \tanh(x)}{2}} \hat{P}_1 + \sqrt{\frac{1 + \tanh(x)}{2}} \hat{P}_2, \quad x \in R. \quad (3)$$

Note that $\hat{P}^2(x) + \hat{P}^2(-x) = \hat{I}$ and therefore $\hat{P}(x)$ and $\hat{P}(-x)$ describe a measurement. If $x = \varepsilon$, where $|\varepsilon| \ll 1$, the measurement is weak. Consider the effect of the operators $\hat{P}(x)$ on a pure state $|\psi\rangle$. The state can be written as $|\psi\rangle = \hat{P}_1|\psi\rangle + \hat{P}_2|\psi\rangle = \sqrt{p_1}|\psi_1\rangle + \sqrt{p_2}|\psi_2\rangle$, where $|\psi_{1,2}\rangle = \hat{P}_{1,2}|\psi\rangle/\sqrt{p_{1,2}}$ are the two possible outcomes of the projective measurement and $p_{1,2} = \langle\psi|\hat{P}_{1,2}|\psi\rangle$ are the corresponding probabilities. If x is positive (negative), the operator $\hat{P}(x)$ increases (decreases) the ratio $\sqrt{p_2}/\sqrt{p_1}$ of the $|\psi_2\rangle$ and $|\psi_1\rangle$ components of the state. By applying the same operator $\hat{P}(\varepsilon)$ many times in a row for some fixed ε , the ratio can be made arbitrarily large or small depending on the sign of ε , and hence the state can be transformed arbitrarily close to $|\psi_1\rangle$ or $|\psi_2\rangle$. The ratio of the p_1 and p_2 is the only parameter needed to describe the state, since $p_1 + p_2 = 1$.

Also note that $\hat{P}(-x)\hat{P}(x) = (1 - \tanh^2(x))^{1/2}\hat{I}/2$ is proportional to the identity. If we apply the same measurement $\hat{P}(\pm\varepsilon)$ twice and two opposite outcomes occur, the system returns to its previous state. Thus we see that the transformation of the state under many repetitions of the measurement $\hat{P}(\pm\varepsilon)$ follows a random walk along a curve $|\psi(x)\rangle$ in state space. The position on this curve can be parameterized by $x = \ln \sqrt{p_1/p_2}$. Then $|\psi(x)\rangle$ can be written as $\sqrt{p_1(x)}|\psi_1\rangle + \sqrt{p_2(x)}|\psi_2\rangle$, where $p_{1,2}(x) = (1/2)[1 \pm \tanh(x)]$.

The measurement given by the operators $\hat{P}(\pm\varepsilon)$ changes x by $x \rightarrow x \pm \varepsilon$, with probabilities $p_{\pm}(x) = (1 \pm \tanh(\varepsilon)(p_1(x) - p_2(x)))/2$. We continue this random walk until $|x| \geq X$, for some X which is sufficiently large that $|\psi(X)\rangle \approx |\psi_1\rangle$ and $|\psi(-X)\rangle \approx |\psi_2\rangle$ to whatever precision we desire. What are the respective probabilities of these two outcomes?

Define $p(x)$ to be the probability that the walk will end at X (rather than $-X$) given that it began at x . This must satisfy $p(x) = p_+(x)p(x + \varepsilon) + p_-(x)p(x - \varepsilon)$. Substituting our expressions for the probabilities, this becomes

$$p(x) = (p(x + \varepsilon) + p(x - \varepsilon))/2 + \tanh(\varepsilon) \tanh(x)(p(x + \varepsilon) - p(x - \varepsilon))/2. \quad (4)$$

If we go to the infinitesimal limit $\varepsilon \rightarrow dx$, this becomes

a continuous differential equation

$$\frac{d^2 p}{dx^2} + 2 \tanh(x) \frac{dp}{dx} = 0, \quad (5)$$

with boundary conditions $p(X) = 1$, $p(-X) = 0$. The solution to this equation is $p(x) = (1/2)[1 + \tanh(x)/\tanh(X)]$. In the limit where X is large, $\tanh(X) \rightarrow 1$, so $p(x) = p_1(x)$. The probabilities of the outcomes for the sequence of weak measurements are exactly the same as those for a single projective measurement. Note that this is also true for a walk with a step size that is not infinitesimal, since the solution $p(x)$ satisfies (4) for an arbitrarily large ε .

Alternatively, instead of looking at the state of the system during the process, we could look at an operator that effectively describes the system's transformation to the current state. This has the advantage that it is state-independent, and will lead the way to decompositions of generalized measurements; it also becomes obvious that the procedure works for mixed states, too.

We think of the measurement process as a random walk along a curve $\hat{P}(x)$ in operator space, given by Eq. (3), which satisfies $\hat{P}(0) = \hat{I}/\sqrt{2}$, $\lim_{x \rightarrow -\infty} \hat{P}(x) = \hat{P}_1$, $\lim_{x \rightarrow \infty} \hat{P}(x) = \hat{P}_2$. It can be verified that $\hat{P}(x)\hat{P}(y) \propto \hat{P}(x + y)$, where the constant of proportionality is $(\cosh(x + y)/2 \cosh(x) \cosh(y))^{1/2}$. Due to normalization of the state, operators which differ by an overall factor are equivalent in their effects on the state. Thus, the random walk driven by weak measurement operators $\hat{P}(\pm\varepsilon)$ has a step size $|\varepsilon|$.

Next we consider measurements where the measurement operators \hat{M}_1 and \hat{M}_2 are *positive* but not projectors. We use the well known fact that a generalized measurement can be implemented as joint unitary operation on the system and an ancilla, followed by a projective measurement on the ancilla [4]. (One can think of this as an *indirect* measurement; one lets the system interact with the ancilla, and then measures the ancilla.) Later we will show that the ancilla is not needed. We consider two-outcome measurements and two-level ancillas. In this case \hat{M}_1 and \hat{M}_2 commute, and hence can be simultaneously diagonalized.

Let the system and ancilla initially be in a state $\rho \otimes |0\rangle\langle 0|$. Consider the unitary operation

$$\hat{U}(0) = \hat{M}_1 \otimes \hat{Z} + \hat{M}_2 \otimes \hat{X}, \quad (6)$$

where \hat{X} and \hat{Z} are Pauli matrices acting on the ancilla bit. By applying $\hat{U}(0)$ to the extended system we transform it to:

$$\begin{aligned} \hat{U}(0)(\rho \otimes |0\rangle\langle 0|)\hat{U}^\dagger(0) = & \hat{M}_1 \rho \hat{M}_1 \otimes |0\rangle\langle 0| \\ & + \hat{M}_1 \rho \hat{M}_2 \otimes |0\rangle\langle 1| \\ & + \hat{M}_2 \rho \hat{M}_1 \otimes |1\rangle\langle 0| \\ & + \hat{M}_2 \rho \hat{M}_2 \otimes |1\rangle\langle 1|. \end{aligned} \quad (7)$$

Then a projective measurement on the ancilla in the computational basis would yield one of the possible generalized measurement outcomes for the system. We can perform the projective measurement on the ancilla as a sequence of weak measurements by the procedure we described earlier. We will then prove that for this process, there exists a corresponding sequence of generalized measurements with the same effect acting solely on the system. To prove this, we first show that at any stage of the measurement process, the state of the extended system can be transformed into the form $\rho(x) \otimes |0\rangle\langle 0|$ by a unitary operation which does not depend on the state.

The net effect of the joint unitary operation $\hat{U}(0)$, followed by the effective measurement operator on the ancilla, can be written in a block form in the computational basis of the ancilla:

$$\begin{aligned}\hat{M}(x) &\equiv (\hat{I} \otimes \hat{P}(x))\hat{U}(0) \\ &= \begin{pmatrix} \sqrt{\frac{1-\tanh(x)}{2}}\hat{M}_1 & \sqrt{\frac{1-\tanh(x)}{2}}\hat{M}_2 \\ \sqrt{\frac{1+\tanh(x)}{2}}\hat{M}_2 & -\sqrt{\frac{1+\tanh(x)}{2}}\hat{M}_1 \end{pmatrix}. \quad (8)\end{aligned}$$

If the current state $\hat{M}(x)(\rho \otimes |0\rangle\langle 0|)\hat{M}^\dagger$ can be transformed to $\rho(x) \otimes |0\rangle\langle 0|$ by a unitary operator $\hat{U}(x)$ which is independent of ρ , then the lower left block of $\hat{U}(x)\hat{M}(x)$ should vanish. We look for such a unitary operator in block form, with each block being Hermitian and diagonal in the same basis as \hat{M}_1 and \hat{M}_2 . One solution is:

$$\hat{U}(x) = \begin{pmatrix} \hat{A}(x) & \hat{B}(x) \\ \hat{B}(x) & -\hat{A}(x) \end{pmatrix}, \quad (9)$$

where

$$\hat{A}(x) = \sqrt{1 - \tanh(x)}\hat{M}_1(\hat{I} + \tanh(x)(\hat{M}_2^2 - \hat{M}_1^2))^{-\frac{1}{2}}, \quad (10)$$

$$\hat{B}(x) = \sqrt{1 + \tanh(x)}\hat{M}_2(\hat{I} + \tanh(x)(\hat{M}_2^2 - \hat{M}_1^2))^{-\frac{1}{2}}. \quad (11)$$

(Since $\hat{M}_1^2 + \hat{M}_2^2 = \hat{I}$, the operator $(\hat{I} + \tanh(x)(\hat{M}_2^2 - \hat{M}_1^2))^{-\frac{1}{2}}$ always exists.) Note that $\hat{U}(x)$ is Hermitian, so $\hat{U}(x) = \hat{U}^\dagger(x)$ is its own inverse, and at $x = 0$ it reduces to the operator (6).

After every measurement on the ancilla, depending on the value of x , we apply the operation $\hat{U}(x)$. Then, before the next measurement, we apply its inverse $\hat{U}^\dagger(x) = \hat{U}(x)$. By doing this, we can think of the procedure as a sequence of generalized measurements on the extended system that transform it between states of the form $\rho(x) \otimes |0\rangle\langle 0|$ (a generalized measurement preceded by a unitary operation and followed by a unitary operation dependent on the outcome is again a generalized measurement). The measurement operators are now $\hat{M}(x, \pm\varepsilon) \equiv \hat{U}(x \pm \varepsilon)(\hat{I} \otimes \hat{P}(\pm\varepsilon))\hat{U}(x)$, and have the form

$$\hat{M}(x, \pm\varepsilon) = \begin{pmatrix} \hat{M}(x, \pm\varepsilon) & \hat{N}(x, \pm\varepsilon) \\ \hat{0} & \hat{O}(x, \pm\varepsilon) \end{pmatrix}. \quad (12)$$

Here $\hat{M}, \hat{N}, \hat{O}$ are operators acting on the system. Upon measurement, the state of the extended system is transformed

$$\rho(x) \otimes |0\rangle\langle 0| \rightarrow \frac{\hat{M}(x, \pm\varepsilon)\rho(x)\hat{M}^\dagger(x, \pm\varepsilon)}{p(x, \pm\varepsilon)} \otimes |0\rangle\langle 0|, \quad (13)$$

with probability

$$p(x, \pm\varepsilon) = \text{Tr} \left\{ \hat{M}(x, \pm\varepsilon)\rho(x)\hat{M}^\dagger(x, \pm\varepsilon) \right\}. \quad (14)$$

By imposing $\hat{M}^\dagger(x, \varepsilon)\hat{M}(x, \varepsilon) + \hat{M}^\dagger(x, -\varepsilon)\hat{M}(x, -\varepsilon) = \hat{I}$, we obtain that

$$\hat{M}^\dagger(x, \varepsilon)\hat{M}(x, \varepsilon) + \hat{M}^\dagger(x, -\varepsilon)\hat{M}(x, -\varepsilon) = \hat{I}, \quad (15)$$

where the operators in the last equation acts on the *system space alone*. Therefore, the same transformations that the system undergoes during this procedure can be achieved by the measurements $\hat{M}(x, \pm\varepsilon)$ acting solely on the system. Depending on the current value of x , we perform the measurement $\hat{M}(x, \pm\varepsilon)$. Due to the one-to-one correspondence with the random walk for the projective measurement on the ancilla, this procedure also follows a random walk with a step size $|\varepsilon|$. It is easy to see that if the measurements on the ancilla are weak, the corresponding measurements on the system are also weak. Therefore we have shown that every measurement with positive operators \hat{M}_1 and \hat{M}_2 , can be implemented as a sequence of weak measurements. This is the main result of this paper. From the construction above, one can find the explicit form of the weak measurement operators:

$$\begin{aligned}\hat{M}(x, \varepsilon) &= \sqrt{\frac{1 - \tanh(\varepsilon)}{2}}\hat{A}(x)\hat{A}(x + \varepsilon) \\ &\quad + \sqrt{\frac{1 + \tanh(\varepsilon)}{2}}\hat{B}(x)\hat{B}(x + \varepsilon). \quad (16)\end{aligned}$$

Note that this procedure works even if the step of the random walk is not small, since $\hat{P}(x)\hat{P}(y) \propto \hat{P}(x+y)$ for arbitrary values of x and y . So it is not surprising that the effective operator which gives the state of the system at the point x is

$$\begin{aligned}\hat{M}(0, x) &= \sqrt{\frac{1 - \tanh(x)}{2}}\hat{M}_1\hat{A}(x) \\ &\quad + \sqrt{\frac{1 + \tanh(x)}{2}}\hat{M}_2\hat{B}(x), \quad (17)\end{aligned}$$

where $\hat{M}(x, y)$ is defined by (16).

Finally, consider the most general type of two-outcome generalized measurement, with the only restriction being $\hat{M}_1^\dagger\hat{M}_1 + \hat{M}_2^\dagger\hat{M}_2 = \hat{I}$. By polar decomposition the measurement operators can be written

$$\hat{M}_{1,2} = \hat{V}_{1,2}\sqrt{\hat{M}_{1,2}^\dagger\hat{M}_{1,2}}, \quad (18)$$

where $\hat{V}_{1,2}$ are appropriate unitary operators. One can think of these unitaries as causing an additional disturbance to the state of the system, in addition to the reduction due to the measurement. The operators $(\hat{M}_{1,2}^\dagger \hat{M}_{1,2})^{1/2}$ are positive, and they form a measurement. We could then measure \hat{M}_1 and \hat{M}_2 by first measuring these positive operators by a sequence of weak measurements, and then performing either \hat{V}_1 or \hat{V}_2 , depending on the outcome.

However, we can also decompose this measurement directly into a sequence of weak measurements. Let the weak measurement operators for $(\hat{M}_{1,2}^\dagger \hat{M}_{1,2})^{1/2}$ be $\hat{M}_p(x, \pm\epsilon)$. Let $\hat{V}(x)$ be any continuous unitary operator function satisfying $\hat{V}(0) = \hat{I}$ and $\hat{V}(\pm x) \rightarrow \hat{V}_{1,2}$ as $x \rightarrow \infty$. We then define $\hat{M}(x, y) \equiv \hat{V}(x+y)\hat{M}_p(x, y)\hat{V}^\dagger(x)$. By construction $\hat{M}(x, \pm y)$ are measurement operators. Since $\hat{V}(x)$ is continuous, if $y = \epsilon$, where $\epsilon \ll 1$, the measurements are weak. The measurement procedure is analogous to the previous cases and follows a random walk along the curve $\hat{M}(0, x) = \hat{V}(x)\hat{M}_p(0, x)$.

In summary, we have shown that for every two-outcome measurement described by operators \hat{M}_1 and \hat{M}_2 acting on a Hilbert space of dimension d , there exists a continuous two-parameter family of operators $\hat{M}(x, y)$ over the same Hilbert space with the following properties: (1) $\hat{M}(x, 0) = \hat{I}/\sqrt{2}$, (2) $\hat{M}(0, x) \rightarrow \hat{M}_1$ as $x \rightarrow -\infty$, (3) $\hat{M}(0, x) \rightarrow \hat{M}_2$ as $x \rightarrow +\infty$, (4) $\hat{M}(x+y, z)\hat{M}(x, y) \propto \hat{M}(x, z+y)$, (5) $\hat{M}^\dagger(x, y)\hat{M}(x, y) + \hat{M}^\dagger(x, -y)\hat{M}(x, -y) = \hat{I}$. We have presented an explicit solution for $\hat{M}(x, y)$ in terms of \hat{M}_1 and \hat{M}_2 . The measurement is implemented as a random walk on the curve $\hat{M}(0, x)$ by consecutive application of the measurements $\hat{M}(x, \pm\epsilon)$, which depend on the current value of the parameter x . In the case where $|\epsilon| \ll 1$, the measurements driving the random walk are weak. Since any measurement can be decomposed into two-outcome measurements, weak measurements are *universal*.

It is obvious from the form of the operators (16) that if a measurement is *local*—the measurement operators $\hat{M}_i \equiv \hat{M}_i \otimes \hat{I}$ act as the identity on all except one subsystem of a composite system—it can be implemented as sequence of weak *local* measurements. This result may be useful for the study of LOCC (Local Operations with Classical Communication).

For example, a very useful concept in the theory of entanglement is the *entanglement monotone* [9], a function of the state that is non-increasing on average under local operations. For pure states the operations are unitaries and generalized measurements. Since all unitaries can be broken into a series of infinitesimal steps and (as we have shown) all measurements can be decomposed into weak measurements, it suffices to look at the behavior of a prospective monotone under small changes in the state. We can thus derive differential conditions for a function to be an entanglement monotone. This was one of the

main motivations for this work.

Moreover, we can think of the set LOCC itself (or at least that subset which preserves pure states) as being generated by infinitesimal local operations. This gives another way of thinking about entanglement protocols, somewhat analogous to studying a Lie algebra by examining the behavior of its generators. These topics will be the subject of a follow-up paper [10].

The connection between weak measurements and quantum trajectories is also an interesting question. Quantum trajectories can be thought of as continuous measurements; these generalized measurements should therefore correspond to continuous measurements where the type of measurement is also continuously adjusted. This might be experimentally feasible for some quantum optical or atomic systems, with possible application to experiments in quantum control. No doubt the decomposition into weak measurements will have many other applications; it adds yet another tool to the arsenal of the quantum information theorist.

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